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On the symmetry of 11-hedra

Yury L. Voytekhovsky[a,b]* and Dmitry G. Stepenshchikov[b]

[a]Laboratory for Mathematical Investigations in Crystallography, Mineralogy and Petrography, High Technologies Centre, Kola Branch of Petrozavodsk State University, 14 Fersman Street, 184209 Apatity, Russia, and [b]Laboratory for Platinum Group Elements Ore Genesis, Geological Institute of the Kola Science Centre, Russian Academy of Sciences, 14 Fersman Street, 184209 Apatity, Russia. Correspondence e-mail: voyt@geoksc.apatity.ru

The symmetry point-group statistics for all combinatorially non-isomorphic 11-hedra (440564 in total) are contributed in the paper for the first time. The most symmetrical shapes with 3 to 36 automorphism group orders (305 in total) are drawn in Schlegel projection and characterized by facet symbols and symmetry point groups.

1. Introduction

The automorphism group orders for 11-hedra with 8 to 13 and 18 vertices were found by Duijvestijn & Federico (1981). Among them, only those with automorphism group orders 1 and 3 can be *a priori* identified as related to 1 and 3 symmetry point groups. The numbers of 11-hedra in each Euler's genera (*i.e.* classes of 11-hedra with the same number of vertices) were given by Engel (1982, 1994). The symmetry point groups for all combinatorially non-isomorphic 4- to 10-hedra and simple (only three edges meet at each vertex) 11- to 13-hedra as well as Schlegel diagrams and facet symbols for the most symmetrical shapes (usually those with automorphism group orders not less than 3) were contributed in our previous papers (Voytekhovsky, 2001*a*; Voytekhovsky & Stepenshchikov, 2002*a*,*b*). Here, we report the symmetry point groups for all non-simple 11-hedra for the first time. For the sake of completeness, the symmetry statistics for the simple 11-hedra are also included in Fig. 1. All the 11-hedra with automorphism group orders not less than 3 are drawn in Schlegel projections and characterized by the facet symbols and symmetry point groups.

2. Generation and characterization of polyhedra

As in previous cases, we generated the polyhedra as their Schlegel projections. This is justified by two theorems: every 3-connected planar graph can be realized as a 3-polyhedron and every combinatorial automorphism of a 3-polyhedron is affinely realizable. That is, there exists to each Schlegel diagram a 3-space realization of a polyhedron such that its edge graph is isomorphic to the Schlegel diagram while its symmetry point group is isomorphic to the automorphism group of the Schlegel diagram.

The diagrams were generated by the Fedorov (1893) recurrence algorithm briefly described by Engel (1994) and Voytekhovsky (2001*b*). As the simple 11-hedra were already found, we used them to generate non-simple polyhedra by the reduction operation w. It is known to reduce any edge joining vertices v_1 and v_2 if all facets containing v_1 but not v_2 have no common vertex with any facet containing v_2 but not v_1. Applying w in all possible ways, we reduced the number of vertices from 18 to 8. The generated shapes were compared and duplicated variants were eliminated. Afterwards, the combinatorially non-isomorphic polyhedra were characterized by the symmetry point groups and facet symbols. The latter shows the numbers of triangular, quadrilateral, pentagonal *etc.* facets in a sequence at a polyhedron.

3. Results and discussion

The automorphism group order and symmetry point-group statistics of 11-hedra with various numbers of vertices are given in Fig. 1. The most symmetrical shapes with automorphism group orders not less than 3 are drawn in Schlegel diagrams in Fig. 2. Unfortunately, in many cases the symmetry point groups of the appropriate polyhedra cannot easily be seen from these diagrams. For example, if a realization of a polyhedron has a facet orthogonal to the main axis then its optimal projection is just onto this facet. Some optimal projections are given in Fig. 3. We work with a computer program to make any diagram so that the symmetry of a polyhedron can immediately be seen.

The facet symbols of the polyhedra are lexicographically ordered below (in brackets) for all Euler's genera. Most of these types of 11-hedra have a realization with $mm2$ symmetry point group except those for which another one is given in parentheses.

V = 8: [10, 1] 1–4.

V = 9: [10, 001] 5–7, 8 ($3m$), [83] 9 (3), 10, 11, 12 ($3m$), 13 ($\bar{6}m2$).

V = 10: [10, 00001] 14, [65] 15–23, [812] 24, 25, [8201] 26.

V = 11: [10, 0000001] 27 ($10mm$), [47] 28–34, [551] 35–36 ($5m$), [632] 37–39, [6401] 40–44, [8021] 45, 46.

V = 12: [29] 47–49, 50–51 ($\bar{6}m2$), [452] 52–57, [4601] 58 (3), 59–61, 62–63 ($3m$), [533] 64 (3), 65–68 ($3m$), [614] 69–71, [6221] 72, 73, [6302] 74, 75, 76 (32), [640001] 77–80, [7031] 81 (3), 82–83 ($3m$), [7300001] 84 ($3m$), [8003] 85 (3), 86, 87 ($\bar{6}m2$), [82000001] 88.

V = 13: [0, 11] 89, 90, [272] 91–94, [2801] 95, 96, [434] 97–100, [4421] 101–104, [4502] 105, [460001] 106, 107, [6041] 108, [6122] 109, 110, [622001] 111, [64000001] 112–114.

a.g.o.	s.p.g.	Vertices											Total
		8	9	10	11	12	13	14	15	16	17	18	
1	1	21	662	5790	24888	63080	102524	110015	78169	35199	9176	970	430494
2	m	12	87	273	646	1076	1520	1648	1449	1034	492	187	8424
	2	1	10	58	72	241	143	375	110	253	24	54	1341
3	3		1			4			5			1	11
4	$mm2$	4	5	13	17	25	26	44	25	40	22	27	248
6	$3m$		2			9			13			5	29
	32					1							1
10	$5m$				2					2			4
12	$\bar{6}m2$		1			3			2			4	10
20	$10mm$				1								1
36	$\overline{18}m2$											1	1
Total		38	768	6134	25626	64439	104213	112082	79773	36528	9714	1249	440564

Figure 1
Automorphism group orders (a.g.o.) and symmetry point groups (s.p.g.) of 11-hedra.

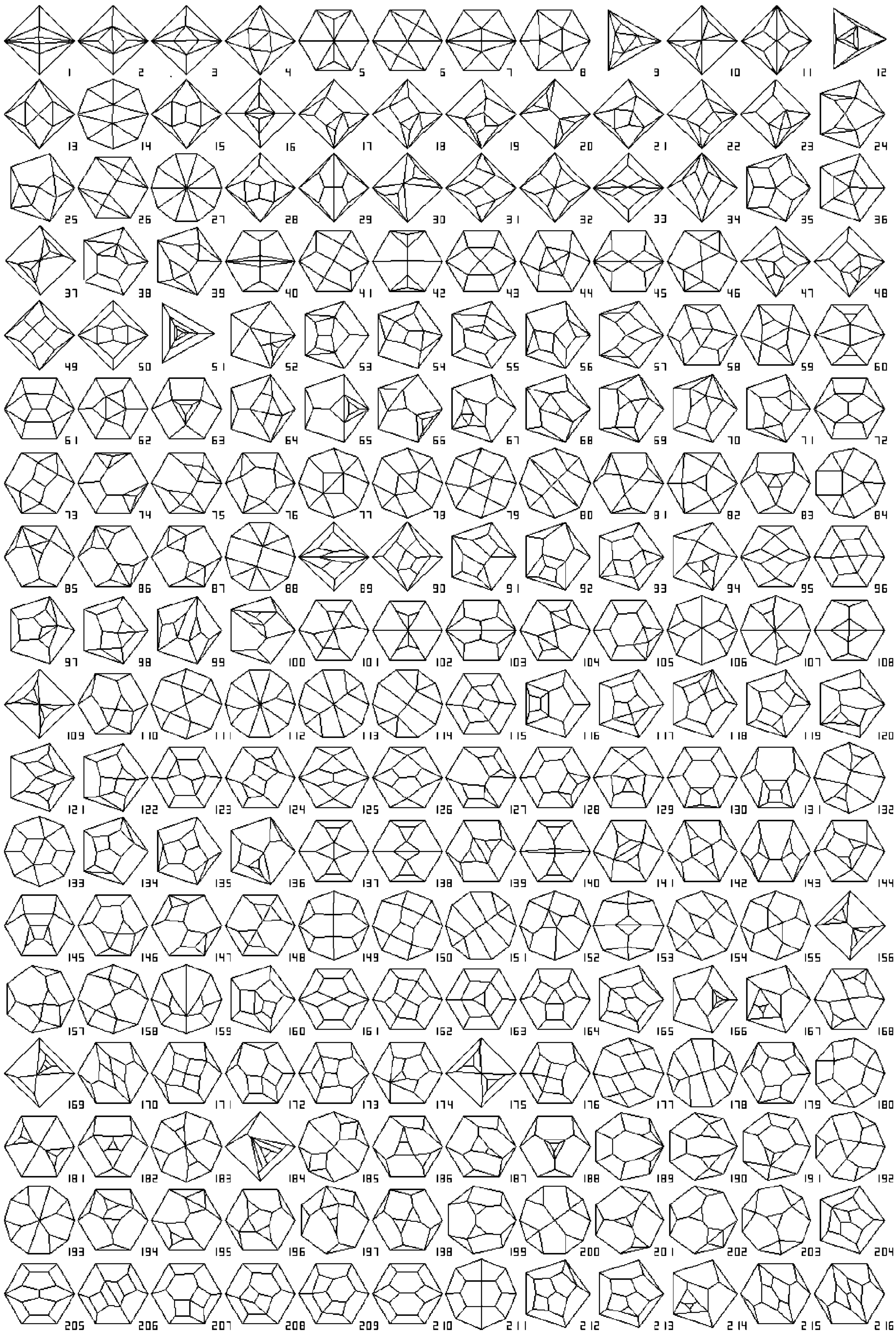

Figure 2
The most symmetrical 11-hedra. See text for facet symbols and symmetries.

V = 14: [0, 10, 01] 115, [092] 116, [254] 117–122, [2621] 123–126, [2702] 127–131, [280001] 132, 133, [416] 134–136, [4241] 137–142, [4322] 143–146, [4403] 147, 148 [442001] 149, 150, [46000001] 151, [604001] 152–155, [6104] 156, [61202] 157, [620201] 158.

V = 15: [0, 10, 0001] 159, [074] 160, [0821] 161, 162, [1631] 163–164 ($3m$), [236] 165, 166 ($3m$), 167 ($\bar{6}m2$), [2441] 168–171, [2522] 172–175, [2603] 176 ($3m$), [262001] 177, [28000001] 178, [3332] 179 (3), [3601001] 180 ($3m$), [4061] 181, 182 ($3m$), [424001] 183–185, [4304] 186 (3), 187–188 ($3m$), [43202] 189–191, [4330001] 192 (3), [44200001] 193, [5033] 194 (3), 195–196 ($3m$), [53003] 197 ($3m$), [6005] 198 ($\bar{6}m2$), [60212] 199, [60400001] 200, [70013] 201 (3), 202 ($3m$), [7003001] 203 ($3m$).

V = 16: [056] 204 ($5m$), [0641] 205, 206, [0722] 207–209, [0803] 210, [082001] 211, [218] 212–214, [2261] 215–218, [2342] 219, 220 [2423] 221, 222 [244001] 223–225, [2504] 226, 227, [25202] 228, 229, [260201] 230, [41402] 231, [4205] 232–236, [42212] 237, [43022] 238, [432002] 239, 240, [44020001] 241, [50500001] 242 ($5m$), [60032] 243, [600401] 244, [602102] 245.

V = 17: [0461] 246, [064001] 247, 248, [07202] 249, 250, [2081] 251, [2162] 252, 253, [226001] 254, [2324] 255, 256, [2405] 257–259, [24212] 260, [25022] 261, [252002] 262, [2700002] 263, [40600001] 264, [41222] 265, [420401] 266, [600221] 267.

V = 18: [0281] 268, [0362] 269, [0443] 270, [0524] 271, [05402] 272, [0605] 273 ($\bar{6}m2$), [06212] 274, [062201] 275, 276, [07022] 277, [072002] 278, 279, [0900002] 280 ($\overline{18}m2$), [1334] 281 ($3m$), [2063] 282, 283 ($\bar{6}m2$), [208001] 284, [2144] 285, [21602] 286, [2225] 287, 288, [224201] 289, [2306] 290, 291 ($\bar{6}m2$), [24032] 292, [242102] 293, [250202] 294, 295, [27000002] 296, [3061001] 297 ($3m$), [40313] 298 ($3m$), [41042] 299, [412202] 300, [420221] 301, [4300301] 302 (3), [430103] 303 ($3m$), [50033] 304 ($3m$), [600203] 305 ($\bar{6}m2$).

The automorphism group order statistics for 11-hedra with 8 to 13 and 18 vertices agree with the data by Duijvestijn & Federico (1981). Such data for 11-hedra of 14 to 17 vertices and the facet symbols as well as symmetry point-group statistics for all 11-hedra are contributed here for the first time. As in the cases of 4- to 10-hedra and simple 12-, 13-hedra (Voytekhovsky, 2001*a*, Voytekhovsky & Stepenshchikov, 2002*a*,*b*), the shapes of 1, m, 2 and $mm2$ symmetry point groups prevail among the 11-hedra. The trivial shapes (of 1 symmetry point group) form the overwhelming majority. The number of polyhedra rapidly drops with increasing symmetry.

4. Conclusions

Up to now, all the varieties of 4- to 11-hedra and simple 12-, 13-hedra have been enumerated and characterized by the facet symbols and symmetry point groups. The most symmetrical shapes of this huge diversity are drawn in Schlegel projections. The next steps are to generate and characterize in the same way all non-simple 12- and simple 14-hedra. However, two intriguing problems still remain almost uninvestigated: how to classify the overwhelming majority of trivial polyhedra and whether some general relations between the combinatorial types (or facet symbols) and symmetry point groups (or automorphism groups) exist or not?

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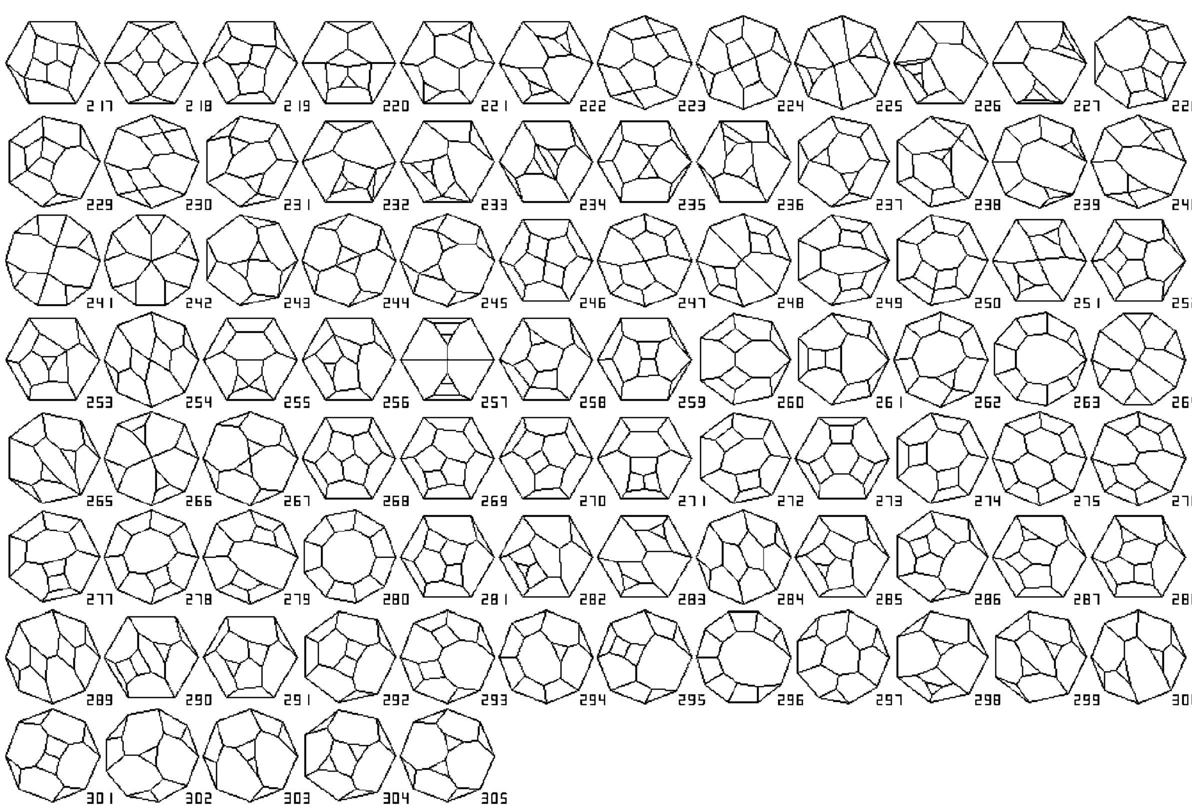

Figure 2 (continued)

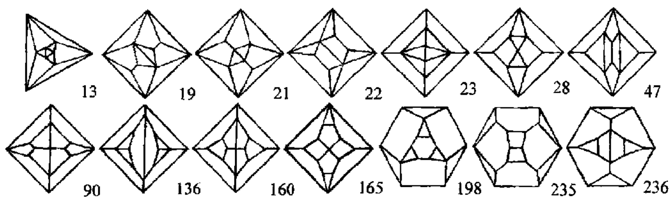

Figure 3
The Schlegel projections of some 11-hedra onto the facet orthogonal to the main symmetry axis. The numbers relate to Fig. 2.

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